Nonnegative matrices in digital signal processing

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Received 7 March 2005; accepted 6 November 2005
Available online 5 December 2005

Abstract

The area of signal processing is very wide. In this review paper we will focus on how the theory of nonnegative matrices may arise in the field of digital filter design: an area in which the role of nonnegative matrices is increasingly relevant due to the development of new technologies in the field of optical components and charge coupling devices.

Keywords: Nonnegative matrices; Digital signal processing; Optical fibers; Charge-coupled devices

1. Introduction

The area of signal processing is very wide. Traditional areas include (as taken from the IEEE Transactions on Signal Processing classification): filter design, fast filtering algorithms, time–frequency analysis, multi-rate filters, signal reconstruction, adaptive filters, nonlinear signals and systems, spectral analysis and so forth. Moreover, the field has been dramatically influenced by related disciplines such as communications and information theory, coding, pattern recognition, image processing, automatic control and identification, just to cite a few. In this large territory nonnegative matrices are often encountered and, in some cases, they are the landowners, as in many areas of information and coding theory, for example.

In this review paper we will focus on how the theory of nonnegative matrices may arise in digital filter design: an area in which the role of nonnegative matrices is becoming increasingly relevant due to the development of new technologies such as optical components and charge coupling devices.

Digital signal processing consists in the representation of signals using sequences of numbers and for data processing tasks. Depending upon the application, such processing may be, for example, data compression and transmission, filtering for smoothing, denoising or for signal reconstruction.

In a general setting, given an input sequence \( u(k) \), \( k = 0, 1, \ldots \), filtering operations require a sequence of actions to be taken in order to obtain a desired output sequence \( y(k) \), \( k = 0, 1, \ldots \). If the requested operations can be accomplished in a linear setting, then the filter, without loss of generality, can be represented by a recurrent difference equation of the form

\[
\begin{align*}
    x(k+1) &= Ax(k) + bu(k), \\
    y(k) &= cx(k) + du(k),
\end{align*}
\]

where \( x(k) \in \mathbb{R}^n \) denotes the state of the filter and \( A \in \mathbb{R}^{n \times n} \), \( b, c^T \in \mathbb{R}^n \), \( d \in \mathbb{R} \) are real matrices with \( n \) being the number of delays.

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The input–output relationship (or transfer function) can be described by means of the $Z$-transform

$$H(z) = \sum_{k \geq 0} h(k)z^{-k} = c(zI - A)^{-1}b + d = N(z)/D(z)$$

(2)

of the impulse response $h(k)$. Moreover, $N(z)$ and $D(z)$ as polynomials in the variable $z$ are such that the order of $D(z)$, called the order of the transfer function, is not less than that of $N(z)$ (proper transfer function). The roots of $D(z)$ are called poles of the transfer function and any pole of maximal modulus is called a dominant pole of $H(z)$. Note that all the poles of $H(z)$ are eigenvalues of the matrix $A$ while the converse in general does not hold.

The realization of such filter requires appropriate devices implementing delays, products and sums, but the physical implementation of these devices may vary very much, depending on the specific technology at hand. Whenever physical constraints imposed by the specific technology in use allow only the implementation of products with nonnegative coefficients, then $A, b, c$ and $d$ in Eq. (1) are nonnegative matrices. As anticipated, this situation occurs in the case of two technologies such as optical fibers and charge-coupled devices. We will briefly describe them hereafter.

Optical fibers provide an attractive technology in view of their low-loss (fractional decibels/kilometer) and large bandwidth-length product (on the order of 100 GHz km for single-mode fibers). In particular, optical filters are of great interest when processing high-speed signals in a waveguide format, thus avoiding costly electro-optic and optoelectronic conversions which eventually create an electronic bottleneck.

When coherence time is less than the shortest delay time in the fiber optic system [1], then an optical filter [7] may process an input signal which is modulated as light intensity variations on optical carriers, by means of optical multipliers, adders and delay lines [1]. In particular, optical multipliers can be realized using optical amplifiers [2] which implement an adjustable positive gain, while optical adders can be made using fiber-optic directional couplers [1] and optical amplifiers. Finally, a delay line of a specified value $\Delta T$ can be implemented using a two-coupler nonrecirculating feed forward delay line as described in [1].

Consequently, since only positive multiplications are allowed, then the matrices appearing in equations (1) are nonnegative.

Charge-coupled technology forms the basis for digital signal processing using charge routing networks (CRNs). This class of filters was introduced at the Bell Labs [3,4] and offers the possibility of achieving discrete-time signal processing on a MOS integrated circuit chip with the advantage of lighter weight, smaller size, lower power consumption and improved reliability (with respect to an equivalent standard implementation).

CRNs are based on a family of functional solid-state electronic devices using MOS technology. Under the application of a sequence of clock pulses, these devices move quantities of electrical charge in a controlled manner across a semiconductor substrate. In particular, a CRN consists of a collection of storage cells, locations where a packet of charge can be stored and maintained isolated from the others, and of a specific periodically repeating routing procedure, operations involving the packets of charge stored in the cells. The basic operations allowed consist in applying a charge packet to a storage cell, in splitting and transferring the charge packet of a cell into distinct cells, in combining charge packets from different cells and transferring them simultaneously into the same cell and in emptying a cell by removing its charge packet from the network while generating a voltage amplitude proportional to the size of the extracted packet. Since the charge can be only splitted in positive quantities, then, also in this case, only nonnegative matrices may appear in Eq. (1).

2. Filter design: the nonnegative realization problem

An important problem in signal processing is filter design, i.e. the physical implementation of a given transfer function $H(z)$ using delays, products and sums. This is nothing but the determination of matrices $A, b, c$ and $d$ such that Eq. (2) holds.

When the matrices allowed by the technology at hand are nonnegative (denoted in the sequel by $A_+, b_+, c_+$ and $d_+$), then, obviously, the impulse response $h(0) = d_+$ and $h(k) = c_+A_+^{k-1}b_+$, for $k > 1$ is nonnegative for all $k \geq 0$. Hence, we can state the design problem called nonnegative (or positive) realization problem as follows [5]: given a proper rational transfer function $H(z)$ with nonnegative impulse response $h(k)$, the quadruple...
\{A_+,b_+,c_+,d_+\} \text{ is said to be a nonnegative realization if }

\[ H(z) = c_+ (z I - A_+)^{-1} b_+ + d_+ \]

with \( A_+,b_+,c_+,d_+ \) nonnegative. The nonnegative realization problem consists of providing answers to the questions:

- **The existence problem**: Is there a nonnegative realization \( \{A_+,b_+,c_+,d_+\} \) of some finite dimension \( N \) and how may it be found?
- **The minimality problem**: What is the minimal value for \( N \)?
- **The generation problem**: How can we generate all possible nonnegative minimal realizations?

It is worth noting that when no specific sign pattern is required for the filter’s matrices, the above problems have a well-known solution: existence is always guaranteed, the minimal order of a realization equals the order of the transfer function and all minimal realizations can be generated by using any invertible change of coordinates. It will be clear in the sequel how nonnegativity dramatically changes this situation leading to an intriguing scenario where the solutions are far from trivial.

In this paper we will focus on the first question giving some introductory examples and main results. The other two questions are not considered here and the interested reader can find a detailed discussion in reference [5] and in the references therein contained.

The geometrical interpretation of the nonnegative realization problem provided in the sequel, requires some basic definitions from cone theory. A set \( \mathcal{K} \subset \mathbb{R}^n \) is said to be a cone provided that \( z \mathcal{K} \subseteq \mathcal{K} \) for all \( z \geq 0 \); if \( \mathcal{K} \) contains an open ball of \( \mathbb{R}^n \) then \( \mathcal{K} \) is said to be solid; if \( \mathcal{K} \cap \{-\mathcal{K}\} = \{0\} \) then \( \mathcal{K} \) is said to be pointed. A cone which is closed, convex, solid and pointed will be called a proper cone. A cone \( \mathcal{K} \) is said to be polyhedral if it is expressible as the intersection of a finite family of closed half-spaces. The notation \( \text{cone}(v_1, \ldots, v_M) \) indicates the closed convex cone consisting of all nonnegative linear combinations of the \( M \) vectors \( v_1, \ldots, v_M \). The following theorem provides a geometrical interpretation of the nonnegative realization problem.

**Theorem 1.** Let \( H(z) \) be a proper rational transfer function of order \( n \) and let \( \{A,b,c,d\} \), with \( A \in \mathbb{R}^{n \times n}, b, c^T \in \mathbb{R}^n \) and \( d \in \mathbb{R} \) be a minimal (i.e. jointly reachable and observable) realization of \( H(z) \). Then, \( H(z) \) has a nonnegative realization if and only if \( d \geq 0 \) and there exists a polyhedral proper cone \( \mathcal{K} \) such that

1. \( A \mathcal{K} \subset \mathcal{K} \), i.e. \( \mathcal{K} \) is \( A \)-invariant;
2. \( \mathcal{K} \subset \mathcal{O} \);
3. \( b \in \mathcal{K} \),

where \( \mathcal{O} = \{x \in \mathbb{R}^n | c A^{k-1} x \geq 0, k = 1,2,\ldots\} \) is called the observability cone.

It is worth noting that, once \( \mathcal{K} = \text{cone}(K) \) has been found, where the columns of \( K \) are the extremal vectors of \( \mathcal{K} \), a nonnegative realization \( \{A_+,b_+,c_+,d_+\} \) can be obtained by solving the set of linear equations

\[
AK = KA_+, \quad b = Kb_+, \quad c_+ = cK, \quad d_+ = d. \quad (3)
\]

Hence, the number of extremal vectors of the cone \( \mathcal{K} \) equals the dimension of the nonnegative realization. This fact amounts to saying that polyhedrality of \( \mathcal{K} \) corresponds to a finite dimension of the nonnegative realization.

First, note that since \( \mathcal{O} = \{x \in \mathbb{R}^n | c A^{k-1} x \geq 0, k = 1,2,\ldots\} \), then it is immediate to see by direct substitution that nonnegativity of the impulse response for \( k > 0 \), \( h(k) = c A^{k-1} b \) is equivalent to the following cone condition:

\[
\mathcal{R} = \text{cone}(b, Ab, A^2 b, \ldots) \subset \mathcal{O},
\]

which defines the reachability cone \( \mathcal{R} \). In view of the geometrical interpretation of the nonnegative realization problem, i.e., Theorem 1, one needs to find a polyhedral proper cone satisfying conditions 1–3. By construction the reachability cone \( \mathcal{R} \) fulfills these conditions apart from polyhedrality, so that it is worth checking polyhedrality of \( \mathcal{R} \).

**Example 2.** Consider the filter with transfer function

\[
H(z) = \frac{2}{z - 1} + \frac{1}{z + 0.4} + \frac{1}{z + 0.8}.
\]

Its impulse response

\[
h(0) = 0, \quad h(k) = 2 + (-0.4)^{k-1} + (-0.8)^{k-1} \quad k \geq 1
\]

is clearly nonnegative for all \( k \). Consider then the minimal realization in Jordan canonical form

\[
A = \text{diag}(1,-0.4,-0.8), \quad b^T = (1 \ 1 \ 1), \quad c = (2 \ 1 \ 1), \quad d = 0.
\]

\[\text{Obviously, the notation } \text{cone}(K) \text{ indicates the cone generated by the columns of the matrix } K.\]
The reachability cone $\mathcal{R}$, as shown on the left-hand side of Fig. 1, is polyhedral with 5 extremal vectors and, in fact,

$$\mathcal{R} := \text{cone}(b, Ab, A^2b, \ldots) = \text{cone}(b, Ab, A^2b, A^3b, A^4b)$$

since $A^5b$ can be expressed as a nonnegative linear combination of vectors $b$, $Ab$, $A^2b$, $A^3b$, and $A^4b.$ For example,

$$A^5b = 0.2304 \cdot b + 0.5696 \cdot Ab + 0.2 \cdot A^3b.$$

A nonnegative realization of order 5 can then be found by solving Eqs. (3) with $K = (b, Ab, A^2b, A^3b, A^4b)$ thus obtaining

$$A_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2304 \\ 1 & 0 & 0 & 0 & 0.5696 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad b_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$c_+ = (4.0.8.2.8.1.4.2.4.2.4.5.2), \quad d_+ = 0.$$

The previous example shows that the reachability cone can play the role of the cone $\mathcal{K}$ in Theorem 1 being polyhedral. Is it always the case? That is, can one always use the reachability cone in order to find a nonnegative realization? The next example illustrates this point.

Example 3. Consider the filter with transfer function

$$H(z) = \frac{1}{z-1} + \frac{1}{z-0.9} + \frac{1}{z-0.8}.$$  

Its impulse response

$$h(0) = 0, \quad h(k) = 1 + (0.9)^{k-1} + (0.8)^{k-1} \quad k \geq 1$$

is clearly nonnegative for all $k.$ Consider then the minimal realization in Jordan canonical form

$$A = \text{diag}(1, 0.9, 0.8), \quad b^T = (1 \ 1 \ 1), \quad c = (1 \ 1 \ 1), \quad d = 0,$$

which, in this case, is a nonnegative realization. Nevertheless, the reachability cone, depicted on the right-hand side of Fig. 1, is nonpolyhedral, i.e., it has an infinite number of extremal vectors. Hence, the reachability cone $\mathcal{R}$ cannot function as the cone $\mathcal{K}$ in Theorem 1, while, in this case, the positive orthant can.

The above example shows that one cannot in general consider only the reachability cone when searching for the cone $\mathcal{K}$ of Theorem 1. Consequently, it is necessary to develop more general methods in order to find such a cone. However, it is worth noting that even if the impulse response is nonnegative, a cone $\mathcal{K}$ satisfying the conditions of Theorem 1 may not exist at all, that is nonnegativity of the impulse response alone is not a sufficient condition for a transfer function to have a nonnegative realization. The next example illustrates this situation.

Example 4. Consider the filter with transfer function

$$H(z) = \frac{1}{z-1} + \frac{z - \cos \varphi}{z^2 - (2 \cos \varphi)z + 1}.$$
whose poles are 1 and $e^{±i\phi}$. Its impulse response

$$h(0) = 0, \quad h(k) = 1 + \cos((k - 1)\phi) \quad k \geq 1$$

is clearly nonnegative for all $k$. Consider then the minimal realization in Jordan canonical form

$$A = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$c^T = \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix}, \quad d = 0.$$  

If $\phi/\pi$ is an irrational number, then neither the reachability cone $\mathcal{R}$ nor any other $A$-invariant proper cone, can be polyhedral. Hence, a nonnegative realization of finite order does not exist.

To show this, suppose there exists an invariant polyhedral proper cone and consider any of its extremal vectors $v$. Since the third component of $v$ remains unchanged under $A$ and the first two components are rotated by an angle $\phi$ in the $x_1$–$x_2$ plane, then it is easily seen that, as $k$ goes to infinity, the cone

$$\text{cone}(v, Av, A^2v, \ldots, A^k v)$$

is an ice-cream cone, thus contradicting the polyhedrality hypothesis.

We conclude the paper by stating the main results on the existence question of the nonnegative realization problem. Note that, when $H(z)$ has only zero poles, then the nonnegative realization problem is trivial since it reduces to the case of pure delays. Hence, in the sequel we will assume that $H(z)$ has a positive dominant pole.

The first result considers the special case of a transfer function with a unique dominant pole:

**Theorem 5.** Let $H(z)$ be a proper rational transfer function of order $n$ with nonnegative impulse response. If $H(z)$ has a unique (possibly multiple) dominant pole, then $H(z)$ has a nonnegative realization of some finite dimension $N \geq n$.

When the transfer function has more than one dominant poles, then the following theorem, which directly follows from the general case described in detail in [5], holds:

**Theorem 6.** Let $H(z)$ be a proper rational transfer function of order $n$ with nonnegative impulse response. Then $H(z)$ has a nonnegative realization of some finite dimension $N \geq n$ if every pole $p_i$ has the property that $p_i/|p_i|$ is a root of unity.

The proofs of the previous theorems (see [5]) are constructive and, consequently, a nonnegative realization can be found (though not necessarily minimal). This means that there is a systematic way to find a filter which can be readily implemented using the technologies described in the Introduction [6].

We conclude by noting that many theoretical and practical problems related to filter design with nonnegative matrices remain open and that we believe this area to be an exciting source for new challenging problems for researchers working in the field of nonnegative matrices.

**References**